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Techniques for finding steady state of circuits with periodic input

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***Abstract*— The paper deals with two techniques used for finding the steady-state response of dynamic circuits excited by periodic input. The first technique is a technique based on the Fourier series. The second technique uses the Laplace transform. Both of these techniques using program composed and run in the MATLAB environment solved the same circuit. The program output were plots of steady state responses to given periodic input in the time domain for periods number of which can be selected by user.**

***Keywords*—Fourier series, Laplace transform, MATLAB, periodic input, steady-state response**

I. INTRODUCTION

Analysis of circuits with periodic input is time-consuming activity. To determine the steady-state response of a circuit excited by a periodic voltage/current source/s it is can be proceeded in two ways. The first way is to use the Fourier series for the input signal representation and then determine the response of the circuit to the dc term, the fundamental, and each harmonic term. Since the circuit is linear one, the principle of superposition is hold, and the steady-state response of the circuit is the sum of the response to the dc term, and each harmonic term [1], [2]. The second way is using the Laplace transform in order to obtain the complex frequency domain representation of the input periodic signal and circuit equation [1], [2], [3]. Then it is necessary to find the periodic solution of given circuit equation and taking the inverse Laplace transform of the obtained solution the steady-state response is found. Both of these techniques can be used in MATLAB environment because MATLAB has required tools. Since the MATLAB environment is very user-friendly, the user has not to spend time with software but he can spend time in learning the fundamental principles of a given problem.

II. THEORETICAL PART OF PROBLEM

Periodic function $f(t)$ is a single-valued function that for every value of time t satisfies the relationship [1], [2]:

$$f(t) = f(t + nT), \quad \text{for } n = \pm 1, \pm 2, \ldots \tag{1}$$

where T is the fundamental period.

A. Fourier series of periodic function

When periodic function $f(t)$ satisfies the Dirichlet's conditions [1], it can be represented by a series consisting of cosine terms (2) or sine terms (3) only, having the integer multiple radian frequencies, with the appropriate coefficients [2]:

$$f(t) = c^{(0)} + \sum_{n=1}^{\infty} c^{(n)} \cos\left(n\omega_0\, t + \varphi^{(n)}\right) \tag{2}$$

or

$$f(t) = c^{(0)} + \sum_{n=1}^{\infty} c^{(n)} \sin\left(n\,\omega_0\, t + \varphi^{(n)} + \frac{\pi}{2} \right) \tag{3}$$

where $\omega_0 = 2\pi/T$ is the fundamental radian frequency, $c^{(0)}, c^{(n)}$ for $n = 1,2,..$ are the Fourier coefficients.

B. Circuit analysis at steady state using the Fourier Series

Let us consider the linear time-invariant dynamic circuit with zero initial conditions, which is excited by a single periodic non-sinusoidal input $f(t)$. Let the transfer functions $H_k(s)$, for $k = 1,2,...,M,$ of the circuit are given as the ratio of the Laplace transform output (circuit element currents and voltages, the number of which is *M*) $X_k(s)$, for $k = 1,2,...,M,$ to the Laplace transform input $F(s)$.

The procedure of circuit analysis using the Fourier series consists of the following steps [1], [2]:

- Calculating the Fourier series representation of the periodic input $f(t)$ (usually truncated to a finite number of components)

$$f(t) = c^{(0)} + \sum_{n=1}^{N} c^{(n)} \cos\left(n\,\omega_0\, t + \varphi^{(n)} \right). \tag{4}$$

- Finding the DC component $x_k^{(0)}$, for $k = 1,2,...,M$, of output, and the *n*-th harmonic component $x_k^{(n)}$, for $k = 1,2,...,M$, of output, for $n = 1,2,...,N$. A linear system having the transfer function $H_k(s)$, responds to a sinusoidal input of frequency $n\omega_0 t$ with the frequency response $\mathbf{H}_k^{(n)}(\mathrm{j}\,n\,\omega_0)$, and the phasor transform method can be applied for finding the individual circuit current and voltage phasor $\mathbf{X}_k^{(n)}$, for $n = 1,2,...,N$,

$$\mathbf{X}_k^{(n)} = \mathbf{H}_k^{(n)}(\mathrm{j}\,n\,\omega_0)\, \mathbf{F}^{(n)}, \quad \text{for } k = 1,2,...,M, \tag{5}$$

  where $\mathbf{H}_k^{(n)}$ is the frequency response function, $\mathbf{F}^{(n)}$ is the phasor of the *n*-th input component $f^{(n)}(t) = c^{(n)} \cos\left(n\,\omega_0\, t + \varphi^{(n)} \right)$.

- Adding up the DC component $x_k^{(0)}$, and the harmonic components $x_k^{(n)}$, for $n = 1,2,...,N$, of output in order to express the total steady-state response $x_{\mathrm{ssF}_k}(t)$

$$x_{\mathrm{ssF}_k}(t) = x_k^{(0)} + \sum_{n=1}^{N} x_k^{(n)}(t) = x_k^{(0)} + \sum_{n=1}^{N} \left| \mathbf{X}_k^{(n)} \right| \cos\left(n\,\omega_0 t + \varphi_k^{(n)} \right), \quad \text{for } k = 1,2,...,M, \tag{6}$$

  where $\varphi_k^{(n)} = \mathrm{tg}^{-1}\left(\mathrm{Im}\left\{ \mathbf{X}_k^{(n)} \right\} \big/ \mathrm{Re}\left\{ \mathbf{X}_k^{(n)} \right\} \right)$, for $k = 1,2,...,M$.

C. The Laplace transform of periodic function

Let $f(t)$ is a periodic input, i.e. a periodic function of real variable *t* having the period $T > 0$ that satisfies the following conditions simultaneously [1], [2]:

- for all $t < 0$: $f(t) = 0$,
- for all $t \geq 0$: $f(t)$ must be piecewise continuous,
- the magnitude of $f(t)$ must be $|f(t)| < \beta \exp(\alpha t)$ for all positive t, where β, α are constants.

Let $f_T(t)$ be a pulse, which is identical to the input $f(t)$ at the time interval $\langle 0, T)$, and zero out of the interval. Repeating the pulse $f_T(t)$ periodically in the time points nT, $n = 0, 1, 2, \ldots$, the input

$f(t)$ is created and it can be expressed for $t \geq 0$ as the sum of infinite number of time-shifted finite pulses [1]:

$$f(t) = \sum_{n=0}^{\infty} f_T(t - nT) \tag{7}$$

Applying the Laplace transform to the equation (7) together with the linearity property and time shifting property of the Laplace transform, the geometric series having a quotient $\exp(-sT)$ is obtained:

$$F(s) = \sum_{n=0}^{\infty} F_T(s).\exp(-s.nT) \tag{8}$$

where $F_T(s)$ is the Laplace transform of the pulse $f_T(t)$.

The geometric series converges for $\left|\exp(-sT)\right| < 1$, i.e. for $\mathrm{Re}\{s\} > 0$, therefore the Laplace transform $F(s)$ of the periodic input $h(t)$ is [1]:

$$F(s) = F_T(s) \big/ \big(1 - \exp(-sT)\big). \tag{9}$$

D. Circuit Analysis at steady state using the Laplace Transform

Let us consider the linear time-invariant dynamic circuit with zero initial conditions, which is excited by a single periodic non-sinusoidal input $f(t)$. Let the transfer functions $H_k(s) = \dfrac{P_k(s)}{Q(s)}$, for $k = 1, 2, ..., M,$ of the circuit are given, thus the circuit element currents and voltages have a form

$$X_k(s) = \frac{P_k(s)}{Q(s)} F(s), \quad \text{for } k = 1, 2, ..., M, \tag{10}$$

where $P_k(s)$ is M_{Pk}-th degree polynomial in *s*, $Q(s)$ is M_Q-th degree polynomials in *s*.

The complex frequency domain representation $X_k(s)$, $k = 1, 2, ..., M$, must have poles at the points, where $\exp(-sT) = 1$, and $Q(s) = 0$. For simplicity, the following assumptions are made for $X_k(s)$, $k = 1, 2, ..., M$:

- The degree of polynomials $P_k(s)$, $k = 1, 2, ..., M,$, are less than that of the polynomial $Q(s)$, i.e. $M_{Pk} < M_Q$.
- All real poles and real parts of all complex poles, which are the roots of $Q(s) = 0$, are negative, so that the circuit is stable.

In order to find the complete response of the circuit to the periodic non-sinusoidal input it is necessary to do the following steps [3]:

Step 1 Finding out the poles of $X_k(s)$, $k = 1, 2, ..., M$, at the points, where $Q(s) = 0$, i.e. finding the roots of the polynomial $Q(s)$.

Step 2 Evaluating $X_k(s)$, $k = 1, 2, ..., M$, but for roots of the polynomial $Q(s)$ only; this part is the transient part $x_{\mathrm{trans}\,k}(t)$ of the complete response of the corresponding circuit element current or voltage for $t > 0$.

Step 3 Taking the inverse Laplace transform of the expression (10) where $F(s)$ is replaced with the terms of $F_T(s)$ that act for $t < \mathrm{T}$, in order to obtain the complete response $x_{T\,k}(t)$, $k = 1, 2, ..., M$, of the corresponding element current or voltage of given circuit for $0 < t < T$, but not for $t > T$.

Step 4 Subtracting the transient response $x_{\mathrm{trans}\,k}(t)$, $k = 1, 2, ..., M$, from the complete response $x_{T\,k}(t)$, $k = 1, 2, ..., M$, of the corresponding circuit element current or voltage at the interval

$0 < t < T$, in order to obtain the steady-state response $x_{ssT\,k}(t)$ of the corresponding circuit element current or voltage at the interval $0 < t < T$

$$x_{ssT\,k}(t) = x_{T\,k}(t) - x_{trans\,k}(t), \quad \text{for } k = 1,2,...,M\,. \tag{11}$$

Step 5 Finding the total steady-state response $x_{ssL\,k}(t)$, $k = 1,2,...,M$, of the corresponding circuit element current or voltage for $t > T$, because the steady-state response $x_{ssT\,k}(t)$, $k = 1,2,...,M$, repeats periodically, i.e.

$$x_{ssL\,k}(t) = \sum_{n=0}^{\infty} x_{ssT\,k}(t - nT)\,, \text{for } k = 1,2,...,M\,. \tag{12}$$

III. Example of Applying Both of Techniques

Let us consider the series *RL* circuit excited by periodic saw-tooth wave input (Fig. 1). The given signal has the following parameters: the amplitude *U* and the period *T*. The aim is to find the steady-state responses of the circuit to given input using the analysis technique based on the Fourier series, and the analysis technique based on the Laplace transform.

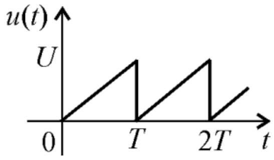

Fig. 1 The circuit input

Given problem was solved in MATLAB environment for the following values of circuit element parameters: $R =$ 2 Ω, $L =$ 0,1 H, $U =$ 50 V, $T =$ 0,05 s.

The corresponding graphs of the steady-state responses of current $i(t)$ (line of red color), the resistor voltage $u_R(t)$ (line of green color), the inductor voltage $u_L(t)$ (line of blue color) for $t \in \langle 0, 2T \rangle$ and the input voltage $u(t)$ (line of black color) , in the case when the circuit simulation was done using the technique based on the Fourier series, truncated to the first five non-zero components, is shown in Fig. 2 (left). In Fig. 2 (right) are depicted the same responses but for case that the Fourier series representation was truncated to the first forty non-zero components.

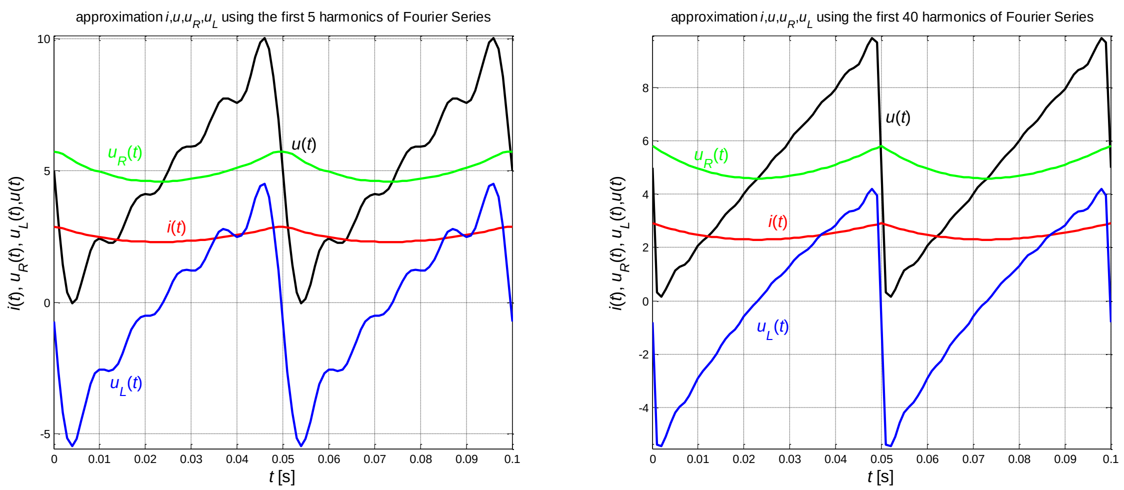

Fig. 2 The steady-state response of the current and voltages using the Fourier series.

The corresponding graphs of the steady-state responses in the case when the circuit simulation was done using the technique based on the Fourier series, truncated to the first hundred non-zero components, is shown in Fig. 3 (left).

It can be seen that the more components are summed in the Fourier series representation (6), the closer their plots become to the solution (12) based on the Laplace transform technique (Fig.3 right).

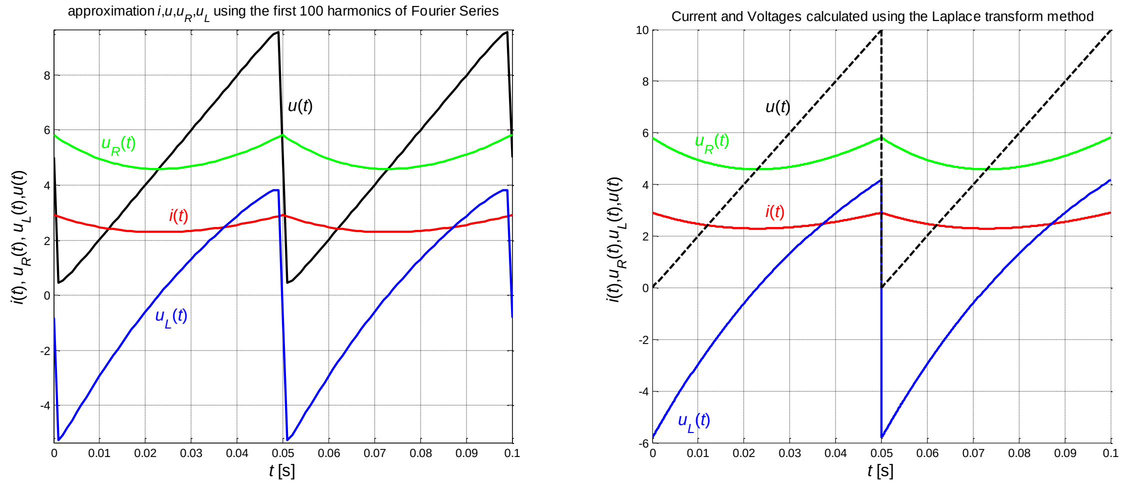

Fig. 3 The steady-state response of the current and voltages using the Fourier series method (left) and Laplace transform method (right).

IV. Conclusions

The paper describes two techniques for finding the steady-state response of the linear circuit to periodic input. The technique using the Fourier series produces the frequency characteristics of the input and output, but not a clear description of the output waveform. The steady-state response is approximated by a finite sum of the Fourier series, and the truncated series causes an error to occur in the output representation. That is a disadvantage of the technique using the Fourier series for finding the steady-state response of the circuit to periodic input.

The technique using the Laplace transform produces a clear description of the output waveform, but no information about the frequency characteristics of the input and output that is a disadvantage of this technique.

It can be said that the Fourier series technique is very useful for analyzing the frequency characteristic of signals and the Laplace transform technique for obtaining analytical description of responses.

References

[1] D. Mayer: *Introduction in Theory of Electric Circuits.* SNTL– Prague, 1984 (in Czech), pp. 503-508.
[2] R.C.Dorf, J.A.Svoboda: *Introduction to Electric Circuits*. John Wiley &Sons, Inc., 2007, pp. 655-700.
[3] I. Mayer: *Circuit Theory (Part 2)*. ALFA - Bratislava, 1981 (in Slovak), pp. 82-90.
[4] MATLAB - *User's Guide* (*Mathematics*, *Programming*, *Graphics*), MathWorks 2009.