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Complete response of underdamped second-order circuits with sinusoidal input

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***Abstract*— The paper deals with finding the complete, natural and steady-state responses of the second-order circuits excited by sinusoidal input in the MATLAB environment. Given problem was solved for various values of circuit element parameters. But only graphical representation of the underdamped circuit responses are mentioned in the paper. In such case the natural and forced responses have the oscillatory-type form, and the interesting phenomena can be observed in the circuit after adding these responses.**

***Keywords*—complete response, MATLAB, natural response, underdamped second-order circuit, sinusoidal input, steady-state response**

I. Introduction

In present days, MATLAB as an interactive software product with powerful built-in routines enables to solve a very wide variety of computations [1] - [4]. Its great advantage is the fact that the MATLAB environment is very user-friendly. Finding the complete response of linear second-order circuit to sinusoidal input can be done in the time domain or in the complex frequency domain, but in the both cases, it is time-consuming activity especially when circuits are complicated.

II. Theoretical Part of Problem

The second-order circuit is a circuit containing two irreducible energy storage circuit elements. It can be a capacitor and inductor, or two irreducible capacitors or two irreducible inductors. In order to find the complete response of the second-order circuit in the time domain we need to make the following steps:

- Represent the circuit by a second-order differential equations,
- Find the general solution of the homogeneous differential equations; this solution we called the natural response or the natural part of the complete response or the transient part of the complete response of the circuit; the natural response of each second-order differential equation will contain two unknown constants,
- Find a particular solution of the differential equations; the particular solution we called the forced response or the forced part of the complete solution or steady-state part of the complete solution of the circuit,
- Represent the complete responses of the second-order circuit as the sum of the natural responses and the forced responses,
- Find and use the initial conditions to evaluate the unknown constants.

In the time domain, the circuits are described by the equations in form of differential equations. A second-order circuit is a circuit that is represented by a second-order differential equation. The order of the differential equation that represents a circuit is equal to the number of irreducible capacitors in the circuit plus the number of irreducible inductors. The second-order differential equation representing a linear second-order circuit takes form [5]:

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + k_1 \frac{\mathrm{d}\, y(t)}{\mathrm{d}t} + k_0 y(t) = x(t) \tag{1}$$

where $y(t)$ is an output of the circuit, $x(t)$ is an input of the circuit (the forcing function that is specified), k_0, k_1 are known constants, and t is the time.

The output $y(t)$ of the circuit, also called the response of the circuit, can be voltage across or current through any circuit element. But we usually choose the voltage/s of the capacitor/s and the current/s through the inductor/ as the output of the circuit. The rest of the element currents and voltages in the given circuit we can be express in the terms of capacitor voltage/s and/or inductor current/s.

The voltages of independent voltage sources and/or currents of independent current sources provide the input $x(t)$ to the circuit.

A. Natural response of the second-order circuit

The natural response $y_{\mathrm{n}}(t)$ of the second-order circuit we found as the general solution of the homogeneous second-order differential equation [5]:

$$\frac{\mathrm{d}^2 y_{\mathrm{n}}(t)}{\mathrm{d}t^2} + k_1 \frac{\mathrm{d}\, y_{\mathrm{n}}(t)}{\mathrm{d}t} + k_0 y_{\mathrm{n}}(t) = 0. \tag{2}$$

The characteristic equation that belongs to the homogeneous second-order differential equation we can readily obtain by replacing the derivative by λ and the second derivative by λ^2 [5]:

$$\lambda^2 + k_1\lambda + k_0 = 0. \tag{3}$$

The roots of the characteristic equation determine the waveform of the natural response of the circuit. They can expressed as

$$\lambda_1, \lambda_2 = -\beta \pm \alpha \tag{4}$$

where $\beta = k_1/2; \alpha = \sqrt{\beta^2 - k_0}$.

When $\beta^2 > k_0$, the quadratic equation (3) has two real and distinct roots:

$$\lambda_1, \lambda_2 = -\beta \pm \alpha. \tag{5}$$

The circuit is said to be overdamped and the natural response $y_{\mathrm{n}}(t)$ of the circuit for $t > 0$ takes form [5]:

$$y_{\mathrm{n}}(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t}, \tag{6}$$

where K_1, K_2 are unknown constants.

When $\beta^2 = k_0$, the quadratic equation (3) has two real equal roots:

$$\lambda_1 = \lambda_2 = -\beta. \tag{7}$$

The circuit is said to be critically damped and the natural response $y_{\mathrm{n}}(t)$ of the circuit for $t > 0$ takes form [5]:

$$y_{\mathrm{n}}(t) = (K_1 + K_2 t)e^{-\beta t}, \tag{8}$$

where K_1, K_2 are unknown constants.

When $\beta^2 < k_0$, the quadratic equation (3) has two complex conjugate roots:

$$\lambda_1, \lambda_2 = -\beta \pm \sqrt{k_0 - \beta^2}. \tag{9}$$

But we usually express these roots in the form

$$\lambda_1, \lambda_2 = -\beta \pm \mathrm{j}\omega_{\mathrm{d}}, \tag{10}$$

where β is called the damping coefficient and ω_{d} is called the damped resonant frequency.

The circuit is said to be underdamped and the natural response $y_{\mathrm{n}}(t)$ of the circuit for $t > 0$ takes form [5]:

$$y_{\mathrm{n}}(t) = K_1 e^{-\beta t} \cos \omega_{\mathrm{d}} t + K_2 e^{-\beta t} \sin \omega_{\mathrm{d}} t, \tag{11}$$

where K_1, K_2 are unknown constants.

B. Forced response of the second-order circuit

The forced response of the second-order circuit can be found as a particular solution of the second-order differential equation (1). When the forcing function $x(t)$ is a sinusoidal function of time t having the amplitude X_{m} and the angular frequency ω_{z}

$$x(t) = X_{\mathrm{m}} \sin \omega_{\mathrm{z}} t, \tag{12}$$

the response $y_{\mathrm{f}}(t)$ to a forcing function will often be of the same form as the forcing function $x(t)$. Thus, we can expect the forced response $y_{\mathrm{f}}(t)$ to be a sinusoidal function [5]:

$$y_{\mathrm{f}}(t) = A \sin \omega_{\mathrm{z}} t + B \cos \omega_{\mathrm{z}} t = Y_{\mathrm{m}} \sin(\omega_{\mathrm{z}} t + \varphi), \tag{13}$$

where *A*, *B* are constant, Y_{m} is the amplitude, ω_{z} is the angular frequency, and φ is the initial phase of the forced response $y_{\mathrm{f}}(t)$.

C. The complete response of the second-order circuit

The complete response $y(t)$ of the second-order circuit is given as the sum of the natural response $y_{\mathrm{n}}(t)$ and the forced response $y_{\mathrm{f}}(t)$, thus [5]:

$$y(t) = y_{\mathrm{n}}(t) + y_{\mathrm{f}}(t) \tag{14}$$

Since the natural response $y_{\mathrm{n}}(t)$ contains two unknown constants K_1, K_2, the complete response $y(t)$ contains these constants too. In order to evaluate the unknown constants we use the initial conditions.

D. Initial conditions

The unknown constants, which we have in the natural and complete response, are determined by the initial conditions.

The initial conditions are the initial and derivative values of the output $y(t)$ of the circuit at time when the disturbance in the circuit was made. If the disturbance in the circuit was made at $t = 0$, then the initial conditions for the output of the circuit are: $y(0_+)$, and its derivative $\dfrac{\mathrm{d}\, y(0_+)}{\mathrm{d}t} = \left.\dfrac{\mathrm{d}\, y(t)}{\mathrm{d}t}\right|_{t=0_+}$.

When the output $y(t)$ of the circuit is the current through the inductor and/or the voltage across the capacitor, we have no problem with finding the corresponding initial conditions. If time, when the disturbance in the circuit was made, is e.g. $t = 0$, then the initial value of the output of the circuit must satisfy an equation $y(0_+) = y(0_-) = y(0)$, because output values must be continuous over time.

The initial values of the derivative of the current through the inductor and/or the derivative of the voltage across the capacitor can be found using the voltage across the inductor $u_L(0_+)$ and the current through the capacitor $i_C(0_+)$ at time $t = 0_+$.

III. Example of Analysis of Second-order Circuit Excited by Sinusoidal Function in MATLAB

Let us consider the second-order circuit (Fig. 1). As long as the switch is in position 1, given circuit is excited by constant voltage source U_1, and we assume that the switch is in position 1 for a long time. Thus, the circuit is at steady state immediately before the throwing the switch. After throwing the switch from position 1 to position 2, the circuit is excited by sinusoidal signal having the following parameters: the amplitude U_{2m} and the angular frequency ω_z. The aim is to find the complete response of the inductor current in given circuit after throwing the switch from position 1 to position 2.

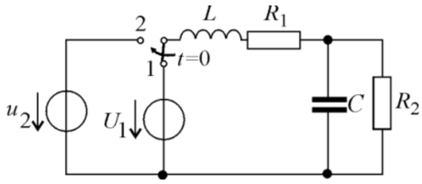

Fig. 1 The circuit input

Given problem was solved in MATLAB environment for various values of circuit element parameters. But we give only responses for the case of underdamped circuit having the following values: U_1 = 12 V, U_{2m} = 24 V, R_1 = 0.02 Ω, R_2 = 199.98 Ω, L = 0.4 mH, C = 100 μF, and the angular frequency ω_z of the voltage source u_2 is:

a) ω_z = 2000 s^{-1} ($\omega_z < \omega_d$),

b) ω_z = 8000 s^{-1} ($\omega_z > \omega_d$),

c) ω_z = 4500 s^{-1} ($\omega_z \approx \omega_d$, but $\omega_z \neq \omega_d$),

d) ω_z = 5000.250019 s^{-1}. ($\omega_z = \omega_d$).

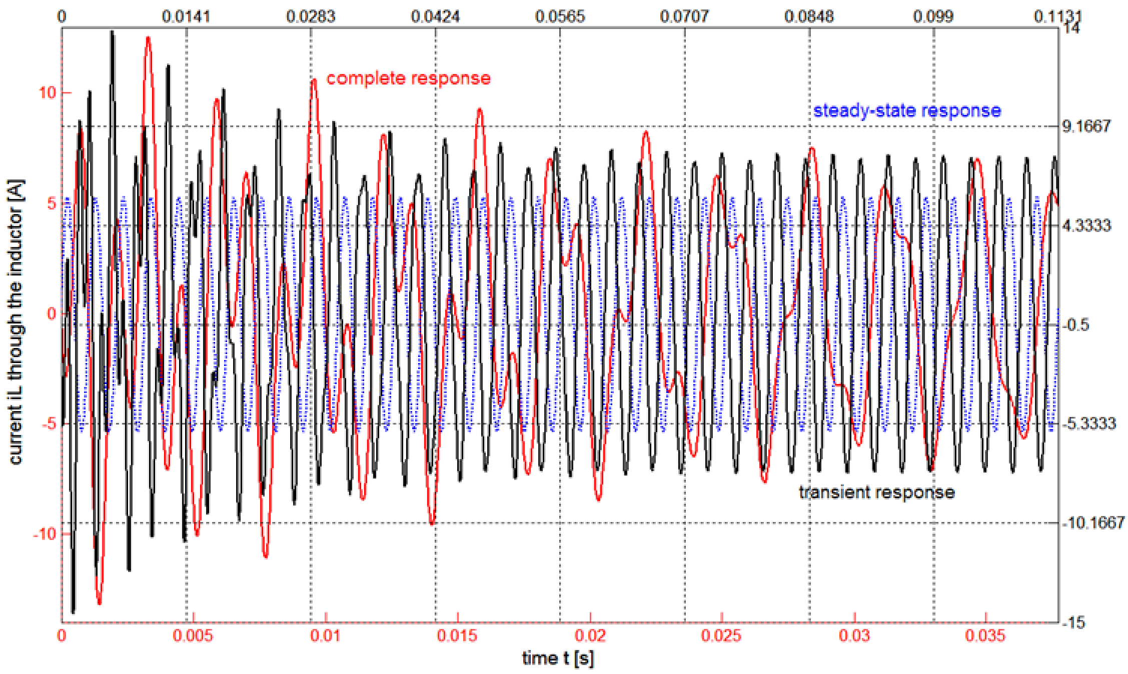

Fig. 2 The complete, transient and steady-state response of the inductor current for $\omega_z < \omega_d$.

The deciding factor for computing the complete response of the underdamped circuit only was the fact that the natural response of the circuit is not a pure oscillatory response, but oscillatory-type response with a decaying magnitude. The forced response of the circuit is a pure oscillatory response, but with a constant magnitude and the frequency of the oscillation ω_z identical to the angular frequency of the forcing function. As a result of composing the natural and forced oscillation, we can observe interesting phenomena in the circuit (damped impacts and an isochronous case), especially when β has a low value, and the damped resonant frequency ω_d is close to or equal to the angular frequency ω_z of the forcing function.

The corresponding graph of the complete, transient and steady-state responses of the inductor current in case a) is shown in Fig. 2.

The graphical representation of the complete, transient and steady-state responses of the inductor current in case a) is shown in Fig. 3.

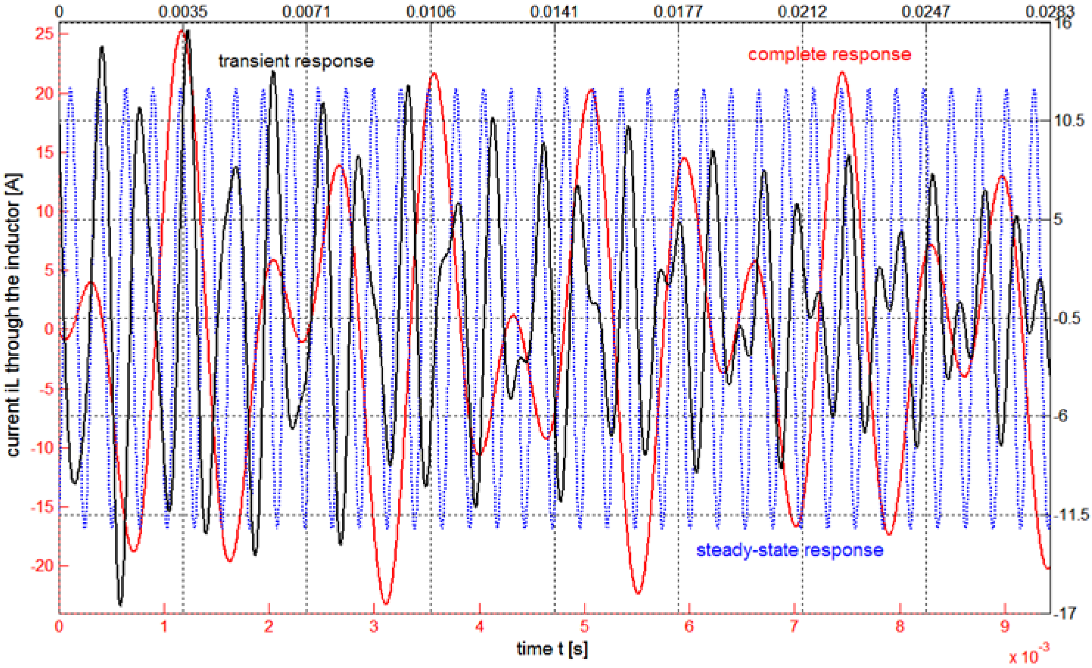

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Fig. 3 The complete, transient and steady-state response of the inductor current for $\omega_z > \omega_d$.

In the case, when $\omega_z \approx \omega_d$, but $\omega_z \neq \omega_d$, and the ratio ω_z / ω_d is nonzero rational number, we can observe damped impacts in the circuit. The complete, transient and steady-state responses of the inductor current in this case are shown in Fig. 4.

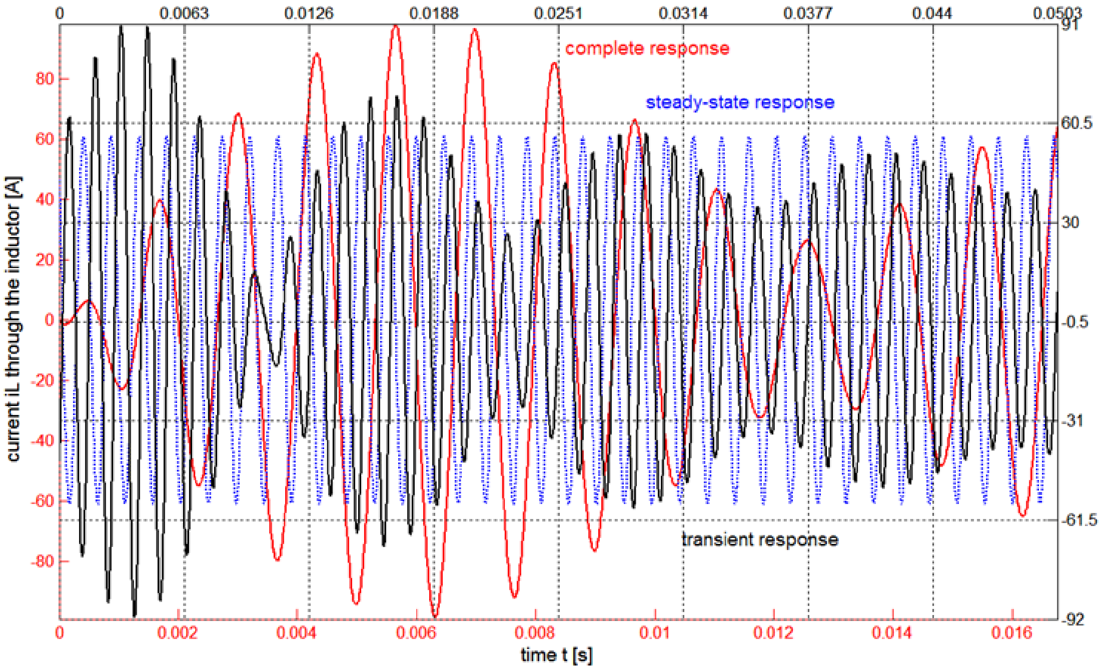

Fig. 4 The complete, transient and steady-state response of the inductor current for $\omega_z \approx \omega_d$.

In the case, when $\omega_z = \omega_d$, we can observe that the circuit is in an isochronous case. The complete, transient and steady-state responses of the inductor current in this case are shown in Fig. 5. An envelope of the current amplitudes is an exponential curve asymptotically approaching to the amplitude value of the inductor current at the steady state.

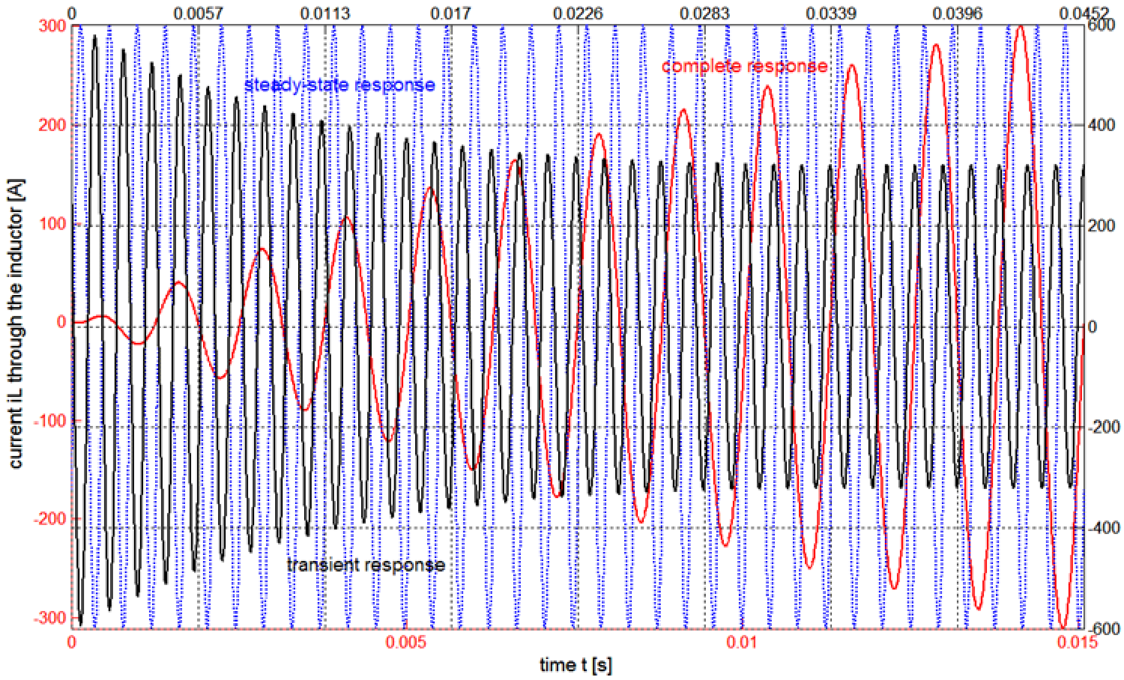

Fig. 5 The complete, transient and steady-state response of the inductor current for $\omega_z = \omega_d$.

The relevant results for drawing the graphs and their visualization were obtained by running program composed in the MATLAB environment.

IV. Conclusions

The paper points out that finding the complete response of underdamped linear second-order circuit and its graphical representation saves a lot of time and effort when the program composed and run in MATLAB environment is used for solution of given problem. MATLAB as an interactive computer program with powerful built-in routines enables to solve a very wide variety of computations. Therefore, everybody who needs to analyze such type of circuits can concentrate on gaining insights, and making conclusions.

References

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